ON AN INTERIOR COMPACTNESS OF ONE HOMOGENEOUS BOUNDARY – VALUE PROBLEM

Rustamova Lamiya Aladdin

Institute of Applied Mathematics, Baku State University, Z. Khalilov 23, AZ1148 Baku, Azerbaijan, e-mail: rustamova@bsu.az

Abstract. *In the paper the conditions are obtained providing existence and uniqueness of the regular solution of the boundary problem for class of the second order homogeneous operator-differential equation with singular coefficients. High term of the equation contains the normal operator the spectrum of which is contained in the certain sectors.*

Further, it is proved the theorem of internal compactness of space of regular solutions of the considered problem.

Keywords *Normal operator, discontinuous coefficients, regular solvability, Hilbert space.*

We consider the boundary problem for homogeneous operator-differential equation in separable Hilbert space$H$

$$-\frac{d^2u}{dt^2}+\rho(t)A^2u+A_0\frac{d^2u}{dt^2}+A_1\frac{du}{dt}+A_2u=0, \tag{1}$$

$$u(0)=\varphi \quad , \tag{2}$$

where $\varphi\in H_{3/2}$, $u(t)\in W_2^2(R_+;H)$, $\rho(t)$ is the form of

$\rho(t)=\left\{\begin{array}{ll}\alpha^2, & t\in(0;1),\\ \beta^2, & t\in(1;\infty)\end{array}\right.$, moreover $\alpha>0$, $\beta>0$, operator coefficients A and A_j $(j=0,1,2)$ satisfy the following conditions

1) A is normal, with quite continuous inverse A^{-1} operator, spectrum of which is contained in a corner sector,

$$S_\varepsilon=\{\lambda:|\arg\lambda|\le\varepsilon\} \quad , \quad 0\le\varepsilon<\pi/2;$$

2) Operators $B_j=A_jA^{-j}$ $(j=0,1,2,)$are bounded in H.

Denote by $L_2(R_+;H)$ a Hilbert space of the vector-functions $f(t)$ with values from H, measurable and integrable by square-law with norm

$$\|f\|_{L_2}=\left(\int_0^{+\infty}\|f(t)\|^2\,dt\right)^{1/2}.$$

Further we introduce the space e.g. [1]

$$W_2^2(R_+;H)=\left\{u(t):\quad u'',A^2u\in L_2(R_+;H)\ ,\ \|u\|_{W_2^2}=\left(\|u''\|_{L_2}^2+\left\|A^2u\right\|_{L_{2,}}^2\right)^{1/2}\right\}.$$

Then from the trace theorem it results that

$$\overset{o}{W}{}_2^2(R_+;H;\{0\}) = \left\{ u \;\middle|\; u \in W_2^2(R_+;H)\,,\, u(0)=0 \right\}.$$

Definition. If for any $\varphi \in H_{3/2}$ there exists the vector-function $u(t)$ which satisfies (1), and boundary condition (2) in the sense

$$\lim_{t\to+0} \|u(t)-\varphi\|_{3/2} = 0$$

also the estimation takes place

$$\|u\|_{W_2^2(R_+;H)} \le const\,\|\varphi\|_{3/2}\,,$$

then $u(t)$ is called a regular solution of the problem (1),(2), and the problem (1),(2) is called regular solvable.

We shall note that when the equations are not homogeneous and $A_0 = 1$, $\alpha = \beta=1$ this problem is investigated in [2], when $A = A^* \ge cE$, $c > 0$, at $A_0 = 1$, $\alpha \ne \beta$ in [3]. When the equation is non homogeneous boundary problem (1),(2) is investigated [4] and resolvability of the equation (1) on all axis it is considered in [5] .

First we shall consider the problem

$$P_o u = -\frac{d^2u}{dt^2} + \rho(t)A^2u = 0 \quad , \tag{3}$$

$$u(0) = \varphi \quad . \tag{4}$$

Let's seek the regular solution of the problem (3), (4) in the form

$$u_0(t) = \begin{cases} e^{-\alpha tA}c_1 + e^{-\alpha(1-t)A}c_2\ , & t\in(0;\ 1), \\ e^{\beta A(1-t)}c_3\ , & t\in(1;\infty), \end{cases}$$

where $c_1\,, c_2\,, c_3$ - are unknown elements from $H_{3/2}$. From the condition (4) and inclusion $u_0(t) \in W_2^2(R_+;H)$ it is obtained the following system of the equations relatively $c_1\,, c_2$ and c_3 :

$$\begin{cases} c_1 + e^{-\alpha A}c_2 = \varphi, \\ e^{-\alpha A}c_1 + c_2 \ = c_3, \\ -\alpha A e^{-\alpha A}c_1 + \alpha A c_2 \ = -\beta A c_3\ , \end{cases}$$

or

$$\begin{cases} c_1 + e^{-\alpha A}c_2 + 0\cdot c_3 = \varphi\ , \\ e^{-\alpha A}c_1 + c_2\ - c_3 = 0\ , \\ -\alpha e^{-\alpha A}c_1 + \alpha c_2\ + \beta c_3\ = 0\ , \end{cases}$$

or in an operational

$$\Delta_0(A)c = \tilde{\varphi},$$

where

$$\Delta_0(A) = \begin{vmatrix} E & e^{-\alpha A} & 0 \\ -e^{-\alpha A} & E & -E \\ -\alpha\, e^{-\alpha A} & \alpha E & \beta E \end{vmatrix},$$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tilde{\varphi} = \begin{pmatrix} \varphi \\ 0 \\ 0 \end{pmatrix}.$$

As we have shown $\Delta_0(A)$ that it is $H^3 = H \times H \times H$(see . [4]), therefore, we shall unequivocally define c_1, c_2 and c_3. They belong $H_{3/2}$, as $\varphi \in H_{3/2}$. It is obvious, that

$$\|u_0(t)\|_{W_2^2(R_+;H)} \leq const\, \|\varphi\|_{3/2}\,.$$

Now we consider the boundary problem (1),(2). For this purpose we take

$$u(t) = \vartheta(t) + u_0(t).$$

Then we get the following equation

$$-\tfrac{d^2\vartheta(t)}{dt^2} + \rho(t)A^2\vartheta(t) + A_0\tfrac{d^2\vartheta(t)}{dt^2} + A_1\tfrac{d\vartheta(t)}{dt} + A_2\vartheta(t) -$$

$$-\ \tfrac{d^2u_0(t)}{dt^2} + \rho(t)A^2u_0(t) + A_0\tfrac{d^2u_0(t)}{dt^2} + A_1\tfrac{du_0(t)}{dt} + A_2u_0(t)\ = 0,$$

$$\vartheta(0) = 0.$$

As

$$-\frac{d^2u_0(t)}{dt^2} + \rho(t)A^2u_0(t) = 0\ \ ,$$

$$-\tfrac{d^2\vartheta(t)}{dt^2} + \rho(t)A^2\vartheta(t) + A_0\tfrac{d^2\vartheta(t)}{dt^2} + A_1\tfrac{d\vartheta(t)}{dt} + A_2\vartheta(t) = g(t) \qquad ,$$

$$\vartheta(0) = 0 \qquad ,$$

where

$$g(t) = -A_0\tfrac{d^2u_0(t)}{dt^2} - A_1\tfrac{du_0(t)}{dt} - A_2u_0(t)\ \ .$$

As operators A_jA^{-j} $(j = \overline{0,2})$ are bounded

$$\|g(t)\|_{L_2(R_+;H)} \le \|A_0\| \left\| \tfrac{d^2u_0}{dt^2} \right\|_{L_2(R_+;H)} +$$

$$+ \left\|A_1A^{-1}\right\| \left\|A\tfrac{du_0}{dt}\right\|_{L_2(R_+;H)} + \left\|A_2A^{-2}\right\| \left\|A^2u_0\right\|_{L_2(R_+;H)}.$$

Applying , the theorem of intermediate derivatives, [1] we have

$$\|g(t)\|_{L_2(R_+;H)} \le const \,\|u_0\|_{W_2^2(R_+;H)} \le const \,\|\varphi\|_{3/2} \;.$$

Thus, we have reduced a boundary problem (1),(2) to the non homogeneous boundary problem with a zero boundary condition. Thus following theorem is valid.

Theorem 1. Let the operator A satisfies the condition 1), but operators $B_j = A_jA^{-j}\ \ (j = 0,1)$ satisfy condition 2) , moreover
where numbers $c_j(\varepsilon;\alpha;\beta)$ are defined as

$$c_0(\varepsilon;\alpha;\beta) = \frac{1}{\min(\alpha^2;\beta^2)} \begin{cases} 1, & 0 \le \varepsilon < \pi/4, \\ \frac{1}{\sqrt{2}\cos\varepsilon}, & \pi/4 \le \varepsilon < \pi/2. \end{cases}$$

$$c_1(\varepsilon;\alpha;\beta) = \frac{1}{2\,\cos\varepsilon\;\min(\alpha;\beta)}, \qquad 0 \le \varepsilon < \pi/2,$$

$$c_2(\varepsilon;\alpha;\beta) = \frac{\max(\alpha;\beta)}{\min(\alpha^2;\beta^2)} \begin{cases} 1, & 0 \le \varepsilon \le \pi/4, \\ \frac{1}{\sqrt{2}\cos\varepsilon}, & \pi/4 \le \varepsilon < \pi/2. \end{cases}$$

Then boundary problem (1), (2) is regularly solvable.

Now we shall study one property of homogeneous regular solutions. Let numbers a, a_1, b_1, b be such, that

$$0 < a < a_1 < b_1 < b < \infty.$$

Denote by $N(P)$ the space of regular solutions of the boundary problem (1), (2). It is obvious, that $N(P)$- linear full subspace in $W_2^2(R_+;H)$. Really, if $u_n(t) \to u(t)$ in $W_2^2(R_+;H)$ and$P\,(d/dt)\,u_n(t) = 0 \quad (u_n(t) \in N(P))$,

$$\|P\,(d/dt)\,(u(t) - u_n(t))\| \le const\,\|u(t) - u_n(t)\| \;\to 0.$$

Then $\|P\,(d/dt)\,u(t)\| = 0$, i.e. $P\,(d/dt)\,u(t) = 0$,hence $u(t) \in N(P)$.
It is obvious, that $N(P) \subset W_2^1(R_+;H)$.

Definition 2. If a, a_1, b_1, b are such, as $0 < a < a_1 < b_1 < b < \infty M > 0$, the set $\left\{u | u \in N(P),\ \|u\|_{W_2^1((a,b);H)} \le M\right\}$ is compact on norm $\|u\|_{W_2^1((a_1,b_1);H)}$we say speak, that space of regular solutions of the problem (1), (2) is internally compact.

We note, that definition of internal compactness for the first time has entered P.D.Laks [6]. At different situations of interior compactness of solutions the

considered works [7, 8]. Following P.D.Laks's [6] work, we have entered concept of interior compactness of the solutions of the homogeneous equations.

Takes place

Theorem 2. A condition of the theorem 1 let satisfied. Then the space of regular solutions of a problem (1), (2) is internally compact.

Proof. Let $0 < a < a_1 < b_1 < b < \infty$ and scalar function $\varphi(t) \in C_0^{\infty}(a, b)$, is such, that

$$\varphi(t) = \begin{cases} 1, & t \in (a_1, b_1), \\ 0, & t \geq b, \ \ t \leq a. \end{cases}$$

Then it is obvious, that for vector functions $\varphi(t)\, u\,(t) \ \in \overset{o}{W}{}_2^2(R_+; H; 0)$ and $u(t) = 0$ at $t \leq a$ and $t \geq b$.

As we have proved, that at performance of conditions of the theorem the following inequality takes place:

$$\left\| P\,(d/dt)\,\varphi u \right\|_{L_2(R_+;H)} \geq const\,\|\varphi u\|_{W_2^2(R_+;H)}$$

For all $u \ \in \overset{o}{W}{}_2^2(R_+; H; 0)$ (see [4], the theorem 2).
From here we have:

$$\left\| -\tfrac{d^2\varphi(t)u(t)}{dt^2} + \rho(t)\varphi(t)A^2u(t) + A_0\tfrac{d^2\varphi(t)u(t)}{dt^2} + A_1\tfrac{d\varphi(t)u(t)}{dt} + A_2\varphi(t)u(t) \right\|_{L_2(R_+;H)} \geq$$
$$\geq const\,\|\varphi u\|_{W_2^2(R_+;H)} \quad .$$

As

$$P\,(d/dt)\,\varphi(t)u(t) = -\tfrac{d^2\varphi(t)u(t)}{dt^2} + \rho(t)\varphi(t)A^2u(t) + A_0\tfrac{d^2\varphi(t)u(t)}{dt^2} +$$
$$+A_1\tfrac{d\varphi(t)u(t)}{dt} + A_2\varphi(t)u(t) = -\tfrac{d^2\varphi(t)}{dt^2}u(t) - 2\tfrac{d\varphi(t)}{dt}\tfrac{du(t)}{dt} - \varphi(t)\tfrac{d^2u(t)}{dt^2} +$$
$$+\rho(t)\varphi(t)A^2u(t) + A_0\left(\tfrac{d^2\varphi(t)}{dt^2}u(t) + 2\tfrac{d\varphi(t)}{dt}\tfrac{du(t)}{dt} + \tfrac{d^2u(t)}{dt^2}\varphi(t)\right) +$$
$$+A_1\left(\tfrac{d\varphi(t)}{dt}u(t) + \tfrac{du(t)}{dt}\varphi(t)\right) + A_2\varphi(t)u(t) =$$

$$= \varphi(t)\left(-\tfrac{d^2u(t)}{dt^2} + \rho(t)A^2u(t) + A_0\tfrac{d^2u(t)}{dt^2} + A_1\tfrac{du(t)}{dt} + A_2u(t)\right) +$$
$$+\left(-\tfrac{d^2\varphi(t)}{dt^2}u(t) - 2\tfrac{d\varphi(t)}{dt}\tfrac{du(t)}{dt} + A_0\left(\tfrac{d^2\varphi(t)}{dt^2}u(t) + 2\tfrac{d\varphi(t)}{dt}\tfrac{du(t)}{dt}\right) + A_1\tfrac{d\varphi(t)}{dt}u(t)\right).$$

As $\varphi(t) = 0$ at $t \geq b$, $t \leq a$ and $u(t)$ - the regular decision,

$$P\,(d/dt)\,u(t) = -\frac{d^2u(t)}{dt^2} + \rho(t)A^2u(t) + A_0\frac{d^2u(t)}{dt^2} + A_1\frac{du(t)}{dt} + A_2u(t) = 0$$

And

$$\|P\,(d/dt)\,u(t)\|_{L_2(R_+;H)} = \left\| -\frac{d^2\varphi(t)}{dt^2}u(t) - 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt} + \right.$$

$$\left. + A_0\left(\frac{d^2\varphi(t)}{dt^2}u(t) + 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt}\right) + A_1\frac{d\varphi(t)}{dt}u(t)\right\|_{L_2(R_+;H)} \geq$$

$$\geq const\,\|\varphi u\|_{W_2^2(R_+;H)} = const\,\|\varphi u\|_{W_2^2((a;b);H)} \geq$$

$$\geq const\,\|\varphi u\|_{W_2^2((a_1;b_1);H)} = const\,\|u\|_{W_2^2((a_1;b_1);H)}\,.$$

Thus,

$$\|u\|_{W_2^2((a_1;b_1);H)} \leq \left\| -\frac{d^2\varphi(t)}{dt^2}u(t) - 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt} + \right.$$

$$\left. + A_0\left(\frac{d^2\varphi(t)}{dt^2}u(t) + 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt}\right) + A_1\frac{d\varphi(t)}{dt}u(t)\right\|_{L_2(R_+;H)} \leq$$

$$\leq const\,\|u(t)\|_{L_2((a;b);H)} + const\left\|\frac{du(t)}{dt}\right\|_{L_2((a;b);H)} + const\,\|A_0u(t)\|_{L_2((a;b);H)} +$$

$$+const\left\|A_0\frac{du(t)}{dt}\right\|_{L_2((a;b);H)} + const\,\|A_1u(t)\|_{L_2((a;b);H)} \leq$$

$$\leq const\left(\|u\|_{L_2((a;b);H)} + \left\|\frac{du(t)}{dt}\right\|_{L_2((a;b);H)} + \|A_0\|\,\|u\|_{L_2((a;b);H)} + \right.$$

$$\left. + \|A_0\|\left\|\frac{du(t)}{dt}\right\|_{L_2^2\,((a;b);H)} + \left\|A_1A^{-1}\right\|\,\|Au\|_{L_2((a;b);H)}\right) \leq const\,\|u\|_{W_2^1((a;b);H)} \leq M.$$

Hence, for anyone $0 < a < a_1 < b_1 < b$we have:

$$\|u\|_{W_2^2((a_1;b_1);H)} \leq const\,\|u\|_{W_2^1((a;b);H)} \ \leq M.$$

As $u \in N(P)$, the set $\left\{u|\,\|u\|_{W_2^2((a_1;b_1);H)}\,, u \in N(P)\right\}$ is limited. From quite continuity of the operator A^{-1} follows, that the space $W_2^2\left((a_1,b_1)\,;H\right)$is enclosed in space $W_2^1\left((a_1,b_1)\,;H\right)$ compactly, i.e.

$$W_2^2\left((a_1,b_1)\,;H\right) \subset W_2^1\left((a_1,b_1)\,;H\right)$$

compactly. Hence, $N(P)$- compact set of century $W_2^1\left((a_1,b_1)\,;H\right)$. Thus, we have proved internal compactness of decisions of a problem (1), (2). The theorem is proved.

References

1. Lions ZH.-L., Madzhenes E. Non homogeneous boundary – value problems and their . M., Mir , 1971,371 p.

2. Mirzoev S.S. The problems to theory to solvability of the boundaryvalue problems for operator -differential of the equations in Hilbert space and spectral problems connected with them. The Thesis on competition dissert. of uch.step.dokt.fiz.-mat. Sciences.Baku 1993, 229 p.

3. Mirzoev S.S., Aliev A.R. On one boundary-value problem for operator-differential equations of the second order with discontinuous coefficient. The Works of the Institute Mat. and Mech. AN ., VI (XIV),Baku, 1997, p.p.117-121.

4. Mirzoyev S.S., Rustamova L.A. On solvability of on boundary-value problem for operator - differential equations of the second order with discontinuous coefficient //An International Journal of Applied and Computational Mathematics, 2006, v. 5, 2, p.191-200.

5. Rustamova L.A. On regular solvability of one class operator - differential equations of the second order. BGU , Vestnik BGU , 2005,1, p.p.43-51.

6. Lax P.D. A Phragmen-Lindelf theorem in harmonic analysis and its application to some questions in the theory of elliptic equations // Comm.Pure Appl.Math., 10 (1957), pp.361- 389.

7. P. Koosis, *Interior compact spaces of functions on a half-line,* Comm. Pure Appl. Math. 10 (1957), pp. 583-615.

8. D. Baranov , Interior-compact subspaces and differentiation in model subspaces, Journal of Mathematical Sciences, Volume 139, Number 2 , 2006, pp. 6369-6373